The Solution of Miller-Ross sequential Fractional Dierential Equations By Sumudu Transform

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Abstract

In this paper, we make use of the Sumudu transform method (STM) a type of Miller-Ross sequential fractional dierential equations. We apply STM to fractional ordinary dierential homogenous equations. We obtain the exact solutions of fractional ordinary dierential homogenous equations in terms of Mittag-Leer functions. Some illustrative examples are also given.

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1 INTRODUCTION

paragraphThe fractional calculus is a generalization of dierentiations and integra-tions to non-integers orders. There are many problems in physics and engineering formulated in terms of fractional dierential and integral equations,such as diusion, signal processing, electrochemistry, viscosity etc. The solution of fractional equa-tions are investigated by many authors using dierent method in obtaining exact and approximate solutions. The Sumudu transform method is applied to obtain'the solution of ordinary dierential equations [7]. The Sumudu transform was rst de-ned by Watugala in 1993,which is used to solve engineering control problems[17]. He extended the Sumudu transform two variables in 2002 [18]. The rst applications to dierential equations and inversion formulae were done by Weerakoon in 1994 and 1998 [15],[16]. The application dealing with the convolution-type integral equations were done by Asiru in 2001,2002 and 2003 [1],[2],[3]. The fundamental properties and applications of Sumudu transform were seen in the paper2006. Moreover the Sumudu transform was also used to solve the fractional dierential equations[5],[6]. In this paper, we can nd an explicit solution of fractional ordinary homogenous dierential equations with Miller-Ross sequential fractional derivative by using the Sumudu transform method.

2 Preliminary Results, Notations and Terminology

In this section we give denitions and some basic results which are used in the paper.

The derivative of the rst order $\frac{d}{dt}$ with derivative of non-integer order D; where 0 1, then consider

$$D^{n} f(t) = \underbrace{DD:::D}_{n\ \text{times}} f(t) \tag{2.1}$$

K.S.Miller and B.Ross called the generalized dierentiation dened by (5:2), where D is the Riemann-Liouville fractional derivative, sequential dierentiation. We consider the fractional derivative of the form [14]

$$D^{n} f(t) = DD:::Df(t) \tag{2.2}$$

where = $_1$ + $_1$ + :::$_n$ are called the sequential derivative and the symbol D is the Sequential dierential operators. The Riemann-Liouville Sequential fractional derivative can be written as [14]

$$ {}_aD_t f(t) = \underbrace{\frac{d}{dt}\frac{d}{dt}:::\frac{d}{dt}}_{n\ \text{times}} {}_aD_t^{(n\ )} f(t); \ (n\ 1\ n) \tag{2.3}$$

and the Caputo Sequential fractional derivative can be written as [14]

$$ {}^c_aD_t f(t) = {}_aD_t^{(n\ )} \underbrace{\frac{d}{dt}\frac{d}{dt}:::\frac{d}{dt}}_{n\ \text{times}} f(t); \ (n\ 1\ n) \tag{2.4}$$

The Sequential fractional derivative can appear in various eld such as physics and applied science, modelling processes. In this chapter we will introduce one of the most important methods of solving linear fractional dierential of the form [14]

$$ {}_aD_t^{m} y(t) + \sum_{k=1}^{m\ 1} P_k(t)\,{}_aD_t^{m\ k} y(t) + P_0(t)y(t) = f(t); \ (0 < t < T < 1\ ) \tag{2.5}$$

where ${}_aD_t^{m}$ represent a general fractional derivative and P_k ; f (t) are function. We will consider an initial -value problem composed of the equation (2.5) and appropri-ate conditions, which depend on a construction of all operators D_k for k 2 f1; :::; mg.

The interesting property of fractional dierential equations which is dierent from its analogy in the theory of ordinary dierential equations. The number of condi-tions is not given by the order of the equation exactly Sequential derivative there exists minimum [$_m$] + 1, but real count is equal to m which depend u on the values $_k$ and not only the order , so that no upper limit.

If fractional Riemann-Liouville derivative, the number of initial conditions is unique but depend on all orders of derivatives appearing in the equation. In case of Caputo fractional derivative coincides with the theory of ordinary dierential equation because its initial conditions are given for integer -order derivatives. For ordinary dierential equation Sumudu transform is a very useful tool. The Sumudu transform method is most powerful methods for solving linear fractional dieren-tial equations with constant coecient. On the other hand it is useless for linear fractional dierentials with general variable coecients or for nonlinear fractional dierential equations.

Denition 2.1. : The Mittag-Leer function was introduced by Mittag-Leer and is denoted by E(z) [9]. It is one parameter generalization of exponential function and is dened as,

$$E(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)} \quad ; 2 C; Re() > 0: \tag{2.6}$$

A two-parameter Mittag-Leer function introduced by R. P. Agarwal [4], denoted by $E_{;}(z)$, is dened as,

$$E_{;}(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k+)} ; > 0; >0: \tag{2.7}$$

Denition 2.2. Consider a set A dened as [17]

$$A = ff(t)j\ 9M; {}_1; {}_2 > 0; jf(t)j \quad Me^{\frac{jtj}{j}} \text{ if } t\ 2\ (1)^j \times [0; 1)g:$$

For all real t 0, the Sumudu transform of a function f (t) 2 A; denoted by S[f(t)] = F(u), is dened as

$$S[f(t)](u) = F(u) = \int_0^{1} e^{t} f(ut)dt; \ u\ 2\ ({}_1; {}_2) \tag{2.8}$$

The function f (t) in equation (2.2) is called the inverse Sumudu transform of F(u) and is denoted by $f(t) = S^{1}[F(u)]$:

Denition 2.3. : The Miller-Ross sequential fractional derivative is dened by [14]

$$\begin{aligned} {}_aD_t^{m} &\ {}_aD_t^{m}\ {}_aD_t^{m\ 1} ::: {}_aD_t^{1} \\ {}_aD_t^{m\ 1} &\ {}_aD_t^{m\ 1}\ {}_aD_t^{m\ 1} ::: {}_aD_t^{1} \end{aligned} \tag{2.9}$$

where

$$m = \sum_{j=1}^{m} {}_j ; \quad 0 < {}_j \ 1:$$

Theorem 2.1. : Let F(u) be the Sumudu transform of the functionf(t), then the Sumudu transform of the Miller-Ross fractional derivative of f (t) of order m, is given by [11]

$$S[{}_0D_t^{m} f(t)](u) = F_m(u) = u^{-m} F(u) - \sum_{k=0}^{m-1} u^{-m-k-1} [{}_0D_t^{m-k-1} f(t)|_{t=0}] \quad (2.10)$$

where

$${}_0D_t^{m-k-1} = {}_0D_t^{m-k-1}\, {}_0D_t^{m-k-1} \cdots {}_0D_t^{-1};\ (k = 0;1;\ldots;m-1)$$

Theorem 2.2. : Let F(u) and G(u) be the Sumudu transforms of f (t) and g(t) respectively. If

$$h(t) = (f(t) * g(t)) = \int_0^t f(\tau)g(t-\tau)d\tau$$

where * denotes convolution of f and g, then the Sumudu transform of h(t) is [6]

$$S[h(t)] = uF(u)G(u): \quad (2.11)$$

Lemma 2.3. : let ;; 2 R and > 0; > 0;n 2 N. Then [11]

$$S\left[t^{n+\beta-1}\left(\frac{\partial}{\partial}\right)^n E_{;}(t)\right] = \frac{1}{(1-u)^{n+1}} : \frac{n!u^{n+}}{+} \quad (2.12)$$

In particular = 0 and Re($\frac{1}{u}$), then [8]

$$S[t^{-1}E_{;}(t)](u) = \frac{u^{-1}}{1-u}: \quad (2.13)$$

3 Homogenous Equations With Sequential Fractional Derivatives:

In general a homogenous linear fractional dierential equation with constant coe-cients and with sequential derivatives has the form

$${}_0D_t^{m} y(t) + \sum_{k=1}^{m-1} A_k(t)\,{}_0D_t^{m-k} y(t) + A_0(t)y(t) = 0 \quad (3.1)$$

where ${}_0D_t^{j}$ for j = 1;2:::; m is the dierent sequential derivatives. It implies a large number of initial conditions in fact independent on m.

$${}_0D_0^{m} y(t)|_{t=0} = b_k \quad (3.2)$$

The general solution of a homogenous linear fractional dierential equation with constant coecients and with sequential derivatives by using Sumudu transform as follows. Applying the Sumudu transform on both sides of (3.1), we obtain

$$S[{}_0D_t^{\beta_m} y(t)](u) + \sum_{k=1}^{m-1} S[A_k(t)\,{}_0D_t^{\beta_k} y(t)](u) + A_0 S[(t)y(t)](u) = 0$$

Using (2.10) and with initial conditions (3.2), we obtain

$$u^{-\beta_m} Y(u) - \sum_{j=1}^{m} u^{-\beta_m + j} b_j + \sum_{k=1}^{m-1} [A_k u^{-\beta_k} Y(u) - \sum_{j=1}^{k} b_j u_{k+j}] + A_0 Y(u) = 0$$

$$\Rightarrow \; Y(u) = \frac{\sum_{k=1}^{m} b_k \sum_{j=k}^{m} u^{-\beta_m + j} \cdots}{u^{-\beta_m} + \sum_{l=1}^{m-1} A_l u^{-\beta_l} + A_0}$$

Taking inverse Sumudu transform on both sides, we get

$$y(t) = S^{-1}\left[\frac{\sum_{k=1}^{m} b_k \sum_{j=k}^{m} u^{-\beta_m + j} \cdots}{u^{-\beta_m} + \sum_{l=1}^{m-1} A_l u^{-\beta_l} + A_0}\right]$$

Example 3.1. Consider the homogeneous fractional dierential equation :

$${}_0D_t^{\frac{4}{3}} y(t) + {}_0D_t^{\frac{1}{2}} y(t) + y(t) = 0 \qquad (3.3)$$

with non-zero initial conditions

$${}_0D_t^{\frac{1}{3}} y(t)|_{t=0} = 1$$
$${}_0D_t^{-\frac{1}{2}} y(t)|_{t=0} = 2$$

Applying Sumudu transform on both sides of (3.3), we get

$$S[{}_0D_t^{\frac{4}{3}} y(t)](u) + S[{}_0D_t^{\frac{1}{2}} y(t)](u) + S[y(t)](u) = 0$$

Using (2.10) and the initial conditions, we obtain

$$u^{-\frac{4}{3}} Y(u) - 1 \quad u^{-\frac{1}{2}} Y(u) - 4 \quad 2u^{-\frac{5}{4}} + 2u^{-\frac{1}{2}} + Y(u) = 0$$

and the assumption

$$\frac{1}{u^{-\frac{4}{3}} Y(u) + 2u^{-\frac{1}{2}}} < 1$$

, then we can write

$$Y(u) = \frac{2u^{-\frac{5}{4}} + 5}{u^{-\frac{4}{3}} + 2u^{-\frac{1}{2}} + 1}$$

Here using the result, we have

$$\frac{1}{u - u} = \sum_{n=0} \frac{u^{n+}}{(1 - u)^{n+1}}$$

Therefore

$$\frac{1}{\;-\;-\;} - \sum (\;1) \frac{u^{(\frac{4}{3})k+\frac{4}{3}}}{\;-\;^{k+1}}$$

ogenous Equations With Riemann-Liouville Frac-tional Derivatives:

In this section, we consider the homogenous linear fractional dierential equation with Riemann-Liouville derivative (2.3) and given initial conditions. Its general form is

$$ {}_0D_m\, y\,(t) + A_k\, {}_0D_t $$
$$ {}_k\, y(t) + A_0 y(t) = 0 $$
$$ k=1 $$
$$ {}_0D_{j\,r}\, y(t)j_{t=0} = b_{j\,r}\ (4.1) $$

where j = 1;:::;m and r = 1;:::; [j].In this section there are more initial con-

ditions that in the initial-value problem (4.1),but the Sumudu transform removes this problem because only integer-order powers of u occur in transformed derivatives next to initial conditions. Applying Sumudu transform on both sides of (4.1), we get

$$ S[{}_0D_t^{m}\, y(t)](u) + \sum_{k=1}^{m\ 1} S[A_k\, {}_0D_t^{k}\, y(t)](u) + A_0 S[y(t)](u) = 0 $$

Using (2.10) and with initial conditions (4.1), we obtain

$$u^{m} Y(u) \quad \sum_{j=1}^{m} u^{m+j} b_j + \sum_{k=1}^{m \quad 1} [A_k u \quad_k Y(u) \quad \sum_{j=1}^{k} b_j u^{k+j}] + A_0 Y(u) = 0$$

$$) \quad Y(u) = \sum_{k=1}^{m} b_k \sum_{j=k}^{m} \frac{u^{j+k}}{u^{m} + \sum_{i=1}^{m \quad 1} A_i u^{i} + A_0}$$

Taking inverse Sumudu transform on both sides, we get

$$y(t) = S^{\quad 1} \sum_{k=1}^{m} b_k \sum_{j=k}^{m} \frac{u^{j+k}}{u^{m} + \sum_{i=1}^{m \quad 1} A_i u^{i} + A_0}$$

Example 4.1. Solve the following initial-value problem where 2 (1;2]; 2 (0;1] and A;B;b_1;b_2;b_3 are real constant

$$D_0 y(t) + AD_0 y(t) + By(t) = 0 \tag{4.2}$$

$$\begin{aligned} &D_0^{\quad 1} y(t)j_{t=0} = b_1 \\ &D_0^{\quad 2} y(t)j_{t=0} = b_2 \\ &D_0^{\quad 1} y(t)j_{t=0} = b_3 \end{aligned} \tag{4.3}$$

Applying Sumudu transform on both sides of (4.2), we get

$$S[D_0 y(t)](u) + AS[D_0 y(t)](u) + BS[y(t)](u) = 0$$

Using (2.10) and the initial conditions, we obtain

$$u \; Y(u) \quad b_2 u^{\quad 1} \quad b_1 + Au \quad Y(u) \quad Ab_3 + BY(u) = 0$$

We see that it possible to combine the corresponding levels of the initial conditions. It is only the rst level initial conditions belonging to the rst derivative of every dierentiating term, Let us denote $D_1 = b_1 + Ab_3$ and $D_2 = b_2$

$$Y(u) = \frac{D_2 u^{\quad 1} + D_1}{u \; + Au \; + B}$$

Here using the result, we have

$$\frac{1}{u \quad u} = \sum_{n=0}^{1} \frac{{}^{n} u^{n+}}{(1 \quad u)^{n+1}}$$

Therefore, we get

$$\frac{1}{u+u^{1}+B} = \sum_{k=0}^{1} \frac{(B)\,(1}{u^{k+}} \frac{u^{k+}}{+Au)^{k+1}}$$

$$= \frac{D_2 u_{+k} \quad 1}{(1+Au\quad)^{k+1}} + \frac{D_1 u_{+k} \quad Y(u)}{(1+Au)^{k+1}}$$

Replacing $=$; $= +k$ and $= +k+1$ in (2.12) and taking inverse Sumudu transform, we get

$$y(t) = \sum^{1} \frac{(B)^{k}}{k!} [D_2 t^{k+\quad 1} E_{;+k}(At\quad) + D_1 t^{k+} E_{;+k+1}(At)]_{k=0}$$

5 Homogenous Equations With Caputo Fractional Deriva-tives:

The Caputo fractional dierential operator can be dened in (2.4) which is dier from fractional derivatives and Riemann-Liouville derivatives because it has a fractional integral in its sequence and all the derivative in the sequence are of the same order. Hence, the number of initial conditions depends on the maximal order of derivative in the equation .Otherwise due to various powers of u next to identical initial conditions in the Sumudu transform of the equation. The general form of the equation is

$$ {}^{c}_{0}D^{m} y(t) + \sum_{k=1}^{m\ 1} A^{c}_{k}\, D_{0}^{\ k} y(t) + A_0 y(t) = 0$$

$$y^{j\quad 1}(t)j_{t=0} = b_j \tag{5.1}$$

where $j = 1;:::;\ [\quad j\ ]$.

Applying Sumudu transform on both sides , we get

$$S[{}^{c}_{0}D^{m} y(t)](u) + \sum_{k=1}^{m\quad 1} A_k S[{}^{c}D_{0}^{\ k} y(t)](u) + A_0 S[y(t)](u) = 0$$

Using (2.10) and with initial conditions (5.1), we obtain

$$u^{m} Y(u) \sum_{j=1}^{m} u^{m+}{}_{j}\, b_j + \sum_{k=1}^{m\ 1} [A_k u\ {}_{k} Y(u) \sum_{j=1}^{k} b_j u^{k+j}] + A_0 Y(u) = 0$$

$$) \quad Y(u) = \sum_{k=1}^{m} b_k \sum_{j=k}^{m} \frac{u^{\quad j\ +\ k}}{u^{\ m} + \sum_{i=1}^{m\ 1} A\,\mu^{\ i} + A_0}$$

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Taking inverse Sumudu transform on both sides, we get

$$y(t) = S\,1 \sum_{k=1}^{m} b_k \sum_{j=k}^{m} \frac{u^{j+k}}{u^{m} + \sum_{i=1}^{m1} A_i u^{i} + A_0}$$

Example 5.1. Solve the following initial-value problem where 2 (1;2]; and A;B;b_1;b_2;b_3 are real constant. 2 (0;1]

$${}^{c}_{0}Dy(t) + A\,{}^{c}_{0}Dy(t) + By(t) = 0 \qquad (5.2)$$

$$y(0) = b_1$$
$$y^{0}(0) = b_2 \qquad (5.3)$$

Applying Sumudu transform on both sides of (5.2) we get

$$S[{}^{c}_{0}Dy(t)](u) + AS[{}^{c}_{0}Dy(t)](u) + BS[y(t)](u) = 0$$

Using (2.10) and the initial conditions (5.3), we obtain

$$u\,Y(u) \quad u^{1}\,y(0) \quad u^{2}\,y^{0}(0) + Au\,Y(u) \quad Au^{1}\,y(0) + BY(u) = 0$$

$$)\ u \quad Y(u)\ u^{1}\ b_1 \quad u^{2}\ b_2 + Au\,Y(u) \quad Au^{1}\ b_1 + BY(u) = 0$$

$$Y(u) = \frac{u^{1}\,b_1 + Au^{1}\,b_1 + u^{2}\,b_2\,u}{+u \quad +B}$$

Here using the result, we have

$$\frac{1}{u \qquad u} = \sum_{n=0}^{1} \frac{{}^{n}u^{n+}}{(1 \quad u)^{n+1}}$$

Therefore, we get

$$\frac{1}{u + Au^{1} + B} = \sum_{k=0}^{1} (\ B) \frac{u^{k+}}{(1 + Au)^{k+1}}$$

$$Y(u) = \sum_{k=0}^{1} (\ B)^{k}\ b_1 \frac{u^{k+1}}{(1 + Au)^{k+1}} + b_2 \frac{u^{k+2}}{(1 + Au)^{k+1}}$$

Replacing = ; = k + 2; and = k + + 2; = k + 3 in (2.12) and taking inverse Sumudu transform, we get

$$y(t) = \sum_{k=0}^{1} \frac{(\ B)^{k}}{k!}\ b_1 t^{k+1} E^{(k)}_{;k+2}(\ At\ ) + A\,t^{k+\ +1} E^{(k)}_{;k+2}(\ At\ )$$

$$+\ b_2 t^{k+2} E^{(k)}_{;k+3}(\ At\ )$$

Conclusion

In this paper, Sumudu transform is successfully applied to solve Miller-Ross se-quential Fractional Dierential Equations. The method used here is highly ecient, powerful and condential tool in terms of nding exact solutions. The solution of Fractional Dierential Equation is obtained in terms of Mittag-Leer function. Also, the Sumudu transform technique is used to solve initial value problems in applied sciences and engineering elds.

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